

5.1

Use mathematical induction in Exercises 3–17 to prove summation formulae. Be sure to identify where you use the inductive hypothesis.

3. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n .

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of a proof that $P(n)$ is true for all positive integers n .

c) What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers n ?

d) What do you need to prove in the inductive step of a proof that $P(n)$ is true for all positive integers n ?

e) Complete the inductive step of a proof that $P(n)$ is true for all positive integers n , identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

14. Prove that for every positive integer n , $\sum_{k=1}^n k 2^k = (n-1)2^{n+1} + 2$.

Use mathematical induction in Exercises 31–37 to prove divisibility facts.

32. Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Exercises 49–51 present incorrect proofs using mathematical induction. You will need to identify an error in reasoning in each exercise.

50. What is wrong with this “proof”?

“Theorem” For every positive integer n , $\sum_{i=1}^n i = (n + \frac{1}{2})^2/2$.

Basis Step: The formula is true for $n = 1$

Inductive Step: Suppose that $\sum_{i=1}^n i = (n + \frac{1}{2})^2/2$.

Then $\sum_{i=1}^{n+1} i = (\sum_{i=1}^n i) + (n + 1)$. By the inductive hypothesis, we have $\sum_{i=1}^{n+1} i = (n + \frac{1}{2})^2/2 + n +$

$1 = (n^2 + n + \frac{1}{4})/2 + n + 1 = (n^2 + 3n + \frac{9}{4})/2 = (n + \frac{3}{2})^2/2 = [(n + 1) + \frac{1}{2}]^2/2$, completing the

inductive step.

5.2

3. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 8$.

11. Consider this variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j$, $4j + 2$, or $4j + 3$ for some nonnegative integer j and the second player wins in the remaining case when $n = 4j + 1$ for some nonnegative integer j .

12. Use strong induction to show that every positive integer can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0=1$, $2^1=2$, $2^2=4$, and so on. [Hint: For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

29. What is wrong with this “proof” by strong induction? “Theorem” For every nonnegative integer n , $5n = 0$. *Basis Step:* $5 \cdot 0 = 0$. *Inductive Step:* Suppose that $5j = 0$ for all nonnegative integers j with $0 \leq j \leq k$. Write $k + 1 = i + j$, where i and j are natural numbers less than $k + 1$. By the inductive hypothesis, $5(k + 1) = 5(i + j) = 5i + 5j = 0 + 0 = 0$.

32. Find the flaw with the following “proof” that every postage of three cents or more can be formed using just 3-cent and 4-cent stamps. *Basis Step:* We can form postage of three cents with a single 3-cent stamp and we can form postage of four cents using a single 4-cent stamp. *Inductive Step:* Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just 3-cent and 4-cent stamps. We can then form postage of $k + 1$ cents by

replacing one 3-cent stamp with a 4-cent stamp or by replacing two 4-cent stamps by three 3-cent stamps.

5.3

3. Find $f(2), f(3), f(4)$, and $f(5)$ if f is defined recursively by $f(0) = -1, f(1) = 2$, and for $n = 1, 2, \dots$

a) $f(n+1) = f(n) + 3f(n-1)$.

b) $f(n+1) = f(n)^2 f(n-1)$.

c) $f(n+1) = 3f(n)^2 - 4f(n-1)^2$.

d) $f(n+1) = f(n-1)/f(n)$.

5. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is a nonnegative integer and prove that your formula is valid.

a) $f(0) = 0, f(n) = 2f(n-2)$ for $n \geq 1$

b) $f(0) = 1, f(n) = f(n-1) - 1$ for $n \geq 1$

c) $f(0) = 2, f(1) = 3, f(n) = f(n-1) - 1$ for $n \geq 2$

d) $f(0) = 1, f(1) = 2, f(n) = 2f(n-2)$ for $n \geq 2$

e) $f(0) = 1, f(n) = 3f(n-1)$ if n is odd and $n \geq 1$ and $f(n) = 9f(n-2)$ if n is even and $n \geq 2$

22. Show that the set S defined by $1 \in S$ and $s + t \in S$ whenever $s \in S$ and $t \in S$ is the set of positive integers.

26. Let S be the set of positive integers defined by Basis step: $1 \in S$.

Recursive step: If $n \in S$, then $3n + 2 \in S$ and $n^2 \in S$.

a) Show that if $n \in S$, then $n \equiv 1 \pmod{4}$.

b) Show that there exists an integer $m \equiv 1 \pmod{4}$ that does not belong to S .

45. Use structural induction to show that $n(T) \geq 2h(T) + 1$, where T is a full binary tree, $n(T)$

equals the number of vertices of T , and $h(T)$ is the height of T .

5.4

3. Trace Algorithm 3 when it finds $\text{gcd}(8, 13)$. That is, show all the steps used by Algorithm 3 to find $\text{gcd}(8, 13)$.

11. Give a recursive algorithm for finding the minimum of a finite set of integers, making use of the fact that the minimum of n integers is the smaller of the last integer in the list and the minimum of the first $n - 1$ integers in the list.

16. Prove that the recursive algorithm for finding the sum of the first n positive integers you found in Exercise 8 is correct.

55. Determine the worst-case complexity of the quick sort algorithm in terms of the number of comparisons used.

Chapter 5—Test 1

1. Use mathematical induction to show that $\sum_{j=0}^n (j+1) = (n+1)(n+2)/2$ whenever n is a nonnegative integer.
2. Show that $3^n < n!$ whenever n is an integer with $n \geq 7$.
3. Suppose that the only currency were 3-dollar bills and 10-dollar bills. Show that every amount greater than 17 dollars could be made from a combination of these bills.
4. Suppose that $\{a_n\}$ is defined recursively by $a_n = a_{n-1}^2 - 1$ and that $a_0 = 2$. Find a_3 and a_4 .
5. Give a recursive algorithm for computing na using addition, where n is a positive integer and a is a real number.

Chapter 5—Test 2

1. What is wrong with the following proof that every positive integer equals the next larger positive integer?
“Proof.” Let $P(n)$ be the proposition that $n = n + 1$. Assume that $P(k)$ is true, so that $k = k + 1$. Add 1 to both sides of this equation to obtain $k + 1 = k + 2$. Since this is the statement $P(k + 1)$, it follows that $P(n)$ is true for all positive integers n .
2. Prove that $\sum_{j=n}^{2n-1} (2j + 1) = 3n^2$ whenever n is a positive integer.
3. Use mathematical induction to show that n lines in the plane passing through the same point divide the plane into $2n$ parts.
4. Let $a_1 = 2$, $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Show that $a_n \leq 3^n$ for all positive integers n .
5. Describe a recursive algorithm for computing 3^{2^n} where n is a nonnegative integer.